Integration by substitution and by parts: practice worksheet

IB Mathematics: analysis and approaches HL · AHL 5.16 · 50 minutes · 42 marks

Question 1 (*no calculator*)

Find $\int x \cos 2x \, \mathrm{d}x$. [3]

Question 2 (*no calculator*)

Using the substitution $u = 1 + x^2$, find the exact value of $\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}} \, \mathrm{d}x$. [4]

Question 3 (*no calculator*)

(a) Find $\int x^2 \sin x \, \mathrm{d}x$. [5]

(b) Hence show that $\int_0^{\pi} x^2 \sin x \, \mathrm{d}x = \pi^2 - 4$. [2]

Question 4 (*no calculator*)

(a) Express $\frac{1}{u(u+1)}$ in partial fractions. [2]

(b) Hence, using the substitution $u = \cos x$, find the exact value of $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x + \cos x} \, \mathrm{d}x$. [5]

Question 5 (*no calculator*)

Let $I = \int e^{2x} \cos x \, \mathrm{d}x$.

(a) Using integration by parts twice, show that $I = \frac{1}{5} e^{2x} (\sin x + 2 \cos x) + c$. [6]

(b) The region R is bounded by the curve $y = e^{2x} \cos x$, the x-axis and the y-axis, for $0 \leq x \leq \frac{\pi}{2}$. Find the exact area of R. [2]

Question 6 (*no calculator*)

A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} kxe^x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a positive constant.

(a) Using integration by parts, show that $k = 1$. [4]

(b) Find $\mathrm{P}(X > \ln 2)$, giving your answer in the form $a + b \ln 2$ where $a, b \in \mathbb{Z}$. [3]

(c) Show that $\mathrm{E}(X) = e - 2$. [4]

(d) The median of X is m. Show that m satisfies the equation $(1 - m)e^m = \frac{1}{2}$. [2]

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Markscheme

Question 1

attempt at integration by parts with $u = x$, $\frac{dv}{dx} = \cos 2x$ **M1**

$\frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx$ **A1**

$\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + c$ **A1**

Award M1 for a correct structure $uv - \int v\,u'$ even with a sign or coefficient error in v. Condone absence of $+c$ until the final A1.

Question 2

attempt to substitute, e.g. $\frac{du}{dx} = 2x$ (or $du = 2x\,dx$) used to change the integrand **M1**

$\frac{1}{2}\int u^{-\frac{1}{2}}\,du$ **A1**

new limits $u = 1$ and $u = 4$ (or returning to x and using original limits) **A1**

$[\sqrt{u}]_1^4 = 2 - 1 = 1$ **A1**

Condone absence of or incorrect limits up to the M1. The third A1 may be earned by writing $\sqrt{1+x^2}$ and substituting $x = 0$, $x = \sqrt{3}$.

Question 3

(a)

attempt at integration by parts with $u = x^2$, $\frac{dv}{dx} = \sin x$ **M1**

$-x^2\cos x + \int 2x\cos x\,dx$ **A1**

second application of parts with $u = 2x$ (or x), $\frac{dv}{dx} = \cos x$ **M1**

$2x\sin x - \int 2\sin x\,dx$ **A1**

$-x^2\cos x + 2x\sin x + 2\cos x + c$ **A1**

Award the first A1 for both the correct uv term and the correct remaining integral; if only one is correct award M1A0. Second M1 requires the correct direction of differentiation (polynomial degree reduced).

(b)

substituting limits into their answer to (a): $\left(-\pi^2(-1) + 0 + 2(-1)\right) - (0 + 0 + 2)$ **M1**

$= (\pi^2 - 2) - 2 = \pi^2 - 4$ **A1 AG**

The A1 requires the values $\cos\pi = -1$, $\sin\pi = 0$, $\cos 0 = 1$ seen or clearly used. Do not award A1 if the answer to (a) is incorrect and the given result is reached by manipulation.

Question 4

(a)

$\frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{u+1}$ and attempt to find A, B (e.g. $1 = A(u+1) + Bu$) **M1**

$\frac{1}{u} - \frac{1}{u+1}$ **A1**

Accept the cover-up method with $A = 1$, $B = -1$ stated.

(b)

$\frac{du}{dx} = -\sin x$ (or $du = -\sin x\,dx$) **A1**

$-\int \frac{1}{u(u+1)}\,du = -\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du$ **M1**

limits $u = 1$ and $u = \frac{1}{2}$ (or equivalent, e.g. $\int_{\frac{1}{2}}^{1}$ with sign reversed) **A1**

$[-\ln u + \ln(u+1)]_1^{\frac{1}{2}}$ or $\left[\ln\frac{u+1}{u}\right]_1^{\frac{1}{2}}$ **A1**

$\ln 3 - \ln 2 = \ln\frac{3}{2}$ **A1**

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Condone absence of or incorrect limits up to the second A1. Condone missing modulus signs. Accept $\ln 3 - \ln 2$ *or* $-\ln\frac{2}{3}$ *for the final A1.*

Question 5

(a)

attempt at integration by parts with $u = e^{2x}$, $\frac{dv}{dx} = \cos x$	**M1**
$I = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx$	**A1**
second application of parts on $\int e^{2x}\sin x\,dx$ with $u = e^{2x}$, $\frac{dv}{dx} = \sin x$	**M1**
$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$	**A1**
$I = e^{2x}\sin x + 2e^{2x}\cos x - 4I$ (recognising the original integral)	**M1**
$5I = e^{2x}(\sin x + 2\cos x) + k$ so $I = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$	**A1**
$I = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$	**AG**

METHOD 2: taking $u = \cos x$, $\frac{dv}{dx} = e^{2x}$ *both times gives* $I = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x - \frac{1}{4}I$*; award marks equivalently. Candidates must use the same pairing in both applications, otherwise the identity* $I = I$ *results and at most M1A1M1A0 is available.*

(b)

area $= \left[\frac{1}{5}e^{2x}(\sin x + 2\cos x)\right]_0^{\frac{\pi}{2}}$	**M1**
$= \frac{1}{5}\left(e^{\pi}(1+0) - (0+2)\right) = \frac{e^{\pi}-2}{5}$	**A1**

Accept $\frac{1}{5}e^{\pi} - \frac{2}{5}$*. Since the curve is non-negative on the interval no modulus is needed.*

Question 6

(a)

attempt at integration by parts on $\int xe^x\,dx$ with $u = x$, $\frac{dv}{dx} = e^x$	**M1**
$\int xe^x\,dx = xe^x - e^x$ $(= (x-1)e^x)$	**A1**
$k\int_0^1 xe^x\,dx = 1$, i.e. $k\left[(x-1)e^x\right]_0^1 = 1$	**M1**
$k\,(0-(-1)) = 1 \Rightarrow k = 1$	**A1**
$k = 1$	**AG**

The second M1 requires the total probability condition with limits 0 and 1 to be explicitly stated. The final A1 requires the values at both limits to be shown.

(b)

$\mathrm{P}(X > \ln 2) = \int_{\ln 2}^1 xe^x\,dx$ or $1 - \int_0^{\ln 2} xe^x\,dx$	**M1**
use of $e^{\ln 2} = 2$: $\left[(x-1)e^x\right]_{\ln 2}^1 = 0 - 2(\ln 2 - 1)$	**A1**
$= 2 - 2\ln 2$	**A1**

Award the first A1 for $(\ln 2 - 1)\cdot 2 + 1 = 2\ln 2 - 1$ *if the complement route is used. Accept* $2(1 - \ln 2)$*.*

(c)

$\mathrm{E}(X) = \int_0^1 x^2e^x\,dx$	**M1**
attempt at integration by parts with $u = x^2$, $\frac{dv}{dx} = e^x$	**M1**
$x^2e^x - 2\int xe^x\,dx = x^2e^x - 2(x-1)e^x$ (using part (a))	**A1**
$\left[x^2e^x - 2(x-1)e^x\right]_0^1 = (e-0) - (0+2) = e-2$	**A1**
$\mathrm{E}(X) = e - 2$	**AG**

Accept a full second application of parts on $\int xe^x\,dx$ *in place of quoting part* (a)*. Accept the antiderivative written as* $e^x(x^2 - 2x + 2)$*.*

(d)

$\int_0^m xe^x \, dx = \frac{1}{2}$, i.e. $[(x-1)e^x]_0^m = \frac{1}{2}$ **M1**

$(m-1)e^m + 1 = \frac{1}{2} \Rightarrow (1-m)e^m = \frac{1}{2}$ **A1**

$(1-m)e^m = \frac{1}{2}$ **AG**

Accept the equivalent condition $\int_m^1 xe^x \, dx = \frac{1}{2}$ *giving* $-(m-1)e^m = \frac{1}{2}$ *directly.*

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